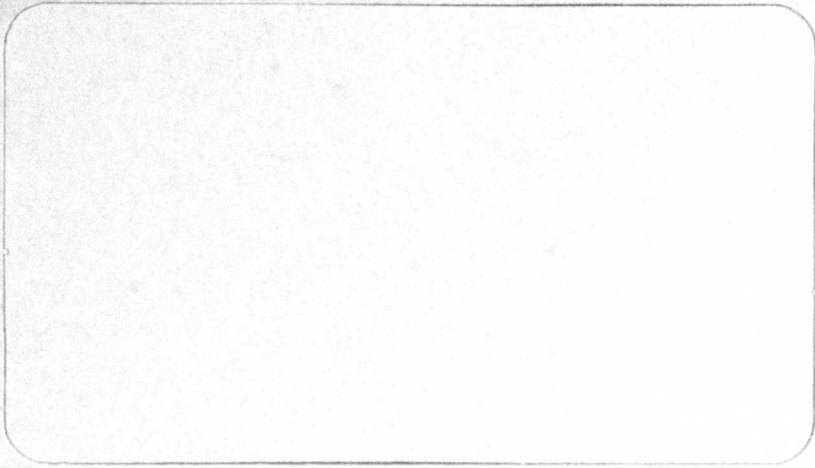


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TRANSIENT FLOWS IN NETWORKS

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SUMMARY

^{the} This paper is concerned with flows in two-terminal dynamic networks as defined by Ford and Fulkerson. These authors have shown how to construct for each positive integer k a flow ϕ_k ^{phi sub k} that maximizes the amount μ_k ^{mu sub k} shipped from source to sink in k time periods. Their method leads to different functions ϕ_k for different values of k . It is here shown that the problem can be solved by means of a single flow $\bar{\phi}$ ^{phi} that at each time k maximizes the cumulative amount shipped from source to sink; the conclusion holds even when the capacities and transit times in the network are allowed to vary with time.

The results of this paper are applicable, for instance, to the analysis of logistics problems involving railway networks. () ←

TRANSIENT FLOWS IN NETWORKS

1. INTRODUCTION

Ford and Fulkerson [1] have introduced the notion of dynamic flows in networks. A dynamic network consists of a graph Γ . Corresponding to each edge e of Γ is a nonnegative integer $\gamma(e)$, called the capacity of the edge, and a second nonnegative integer $\tau(e)$, called the transit time of the edge. In terms of transportation networks, the capacity γ is to be thought of as giving an upper bound to the amount that can be shipped along an edge e , while the transit time τ specifies how long it takes a shipment to traverse an edge e . In this framework, the above authors have considered the following question: Let Γ be a dynamic network, with two distinguished terminals s and s' , called the source and the sink, respectively. Problem: to determine the maximum amount μ_k that can be shipped from s to s' in k time periods. In the work referred to, the authors describe an ingenious algorithm for obtaining μ_k for each integer k . More precisely, they show for each integer k how to obtain a flow ϕ_k , to be thought of as a shipping schedule, that achieves the desired shipment μ_k from s to s' .

Concerning the solution of Ford and Fulkerson, the following observation may be made. In order to achieve the maximum numbers, $\mu_1, \mu_2, \dots, \mu_k$, the authors construct a sequence of flows $\phi_1, \phi_2, \dots, \phi_k$. It would be computationally

advantageous if it turned out that ϕ_2 was a "continuation" of ϕ_1 and, in general, ϕ_{i+1} a continuation of ϕ_i . Put another way, one might hope that the flow ϕ_k had the property that for each time $i < k$ the amount already shipped into s' would be the maximum μ_i . In this way, the single flow ϕ_k would provide a solution to the maximum problem, not only for k time periods, but also for any smaller number of periods. However, the flows obtained by the authors do not have this desirable property; indeed, it is not clear from their work that such universal maximal flows exist. It is our purpose here to show that they do, not only for the case treated by Ford and Fulkerson, but also for the considerably more general case in which the capacities γ and transit times τ may vary with time.

A verbal example may help to clarify further what we intend to show. Suppose one is concerned with shipment by rail of automobiles from Detroit to Los Angeles, and suppose --- in view of the known schedules of freight trains --- it is ascertained that separate shipping schemes can be arranged so that each of the following is possible:

- 20 cars arrive in Los Angeles by Monday,
- 30 cars arrive by Tuesday,
- 70 cars arrive by Wednesday,
- 105 cars arrive by Thursday.

Our theorem applied to this situation asserts that it is then possible to ship in such a way that 20 cars are delivered on Monday, 10 more on Tuesday, 40 more on Wednesday, and the

last 35 on Thursday.

2. A LEMMA ON STATIC NETWORK FLOWS

The result needed for proving the main theorem of this paper (see Sec. 3) is the Feasibility Theorem obtained by the author in [2]. We shall here record the definitions needed for a statement of that result. For motivation and interpretation of these definitions, the reader is referred to [2].

A network with a source is a triple $[X, s; \gamma]$, where X is a finite set of elements x, y, \dots , called nodes; s is a distinguished node of X called the source; and γ , the capacity of the network, is a function on pairs (x, y) of nodes, such that $\gamma(x, y)$ is a nonnegative integer or plus infinity.

A flow ϕ on X is a function from pairs (x, y) to the integers satisfying the conditions

$$\phi(x, y) + \phi(y, x) = 0 \quad (\text{skew-symmetry}),$$

$$\phi(x, y) \leq \gamma(x, y) \quad (\text{feasibility}).$$

A demand δ is a function from $X - s$ to nonnegative integers. The demand δ is called feasible if there exists a flow ϕ such that

$$\delta(y) \leq \sum_{x \in X} \phi(x, y)$$

for all y in $X - s$.

The main result of [2] states the following:

Feasibility Theorem. The demand δ is feasible if and only if, for every subset S of $X - s$, δ satisfies the relation

$$(1) \quad \sum_{y \in S} \delta(y) \leq \sum_{\substack{x \in X-S \\ y \in S}} \gamma(x, y).$$

We shall need a simple corollary of this theorem.

Lemma 1. Let y_1, \dots, y_n be distinct nodes of $X - s$, and let $\delta_1, \dots, \delta_n$ be feasible demands such that $\delta_1(y_1) = \mu_1 \leq \delta_{1+1}(y_{1+1}) = \mu_{1+1}$ for $1 < n$. Let δ be the demand such that

$$\delta(y_1) = \mu_1,$$

$$\delta(y_i) = \mu_i - \mu_{i-1} \quad \text{for } i > 1,$$

$$\delta(y) = 0 \quad \text{otherwise.}$$

Then the demand δ is feasible.

Proof. Let S be any subset of $X - s$, and let k be the largest index for which y_k belongs to S . Then

$$(2) \quad \sum_{y \in S} \delta(y) \leq \sum_{i \leq k} \delta(y_i) = \mu_k;$$

but since δ_k is feasible, it follows from (1) that

$$(3) \quad \mu_k = \delta_k(y_k) \leq \sum_{y \in S} \delta_k(y) \leq \sum_{\substack{x \in X-S \\ y \in S}} \gamma(x, y).$$

Combining (2) and (3), we see that δ satisfies (1) and is therefore feasible.

3. THE TRANSIENT FLOW THEOREM

As mentioned in the Introduction, we intend to consider a generalization of the Ford-Fulkerson dynamic network in which capacities and transit times are allowed to vary with time. This generalization would seem to be useful in terms of applications. In rail networks, for example, it may happen that trains travel on some routes only on certain days of the week, so that the capacity of such a route is sharply increased on these occasions. Likewise, certain routes may sometimes be closed for periodic inspection, etc. We shall therefore give a slightly different formulation of a dynamic network from that of [1].

A two-terminal network $[X, s, s'; \gamma]$ is a network in the sense of the preceding section, having an additional distinguished node s' , called the sink. For brevity, we shall henceforth denote this network simply by X .

A maximal demand δ on such a network is a feasible demand δ for which the value $\delta(s')$ is as large as possible.

Now, let X be the set of nodes of a two-terminal network. We define X_n to consist of all pairs (x, i) , where x is in X and $1 \leq i \leq n$ is a nonnegative integer. We shall, for convenience, denote such a pair by x_i .

An n -stage two-terminal network is a network $[X_n, s_0, s'_n; \gamma]$, where, as before, γ is a function on pairs (x_i, y_j) into non-negative integers or infinity, which must also satisfy

$$(4) \quad \gamma(s_1, s_{1+1}) = \gamma(s'_1, s'_{1+1}) = \infty.$$

For brevity, we henceforth denote this network by X_n .

An explanatory word concerning this definition is in order. The number $\gamma(x_i, y_j)$ gives an upper bound to the amount that can be shipped from node x at time i and arrive at node y at time j . In view of this interpretation, one might expect the condition $\gamma(x_i, y_j) = 0$ for $j \leq i$ (the irreversibility of time). However, since our argument is independent of this condition, there is no reason to impose it. Condition (4), above, states that goods can always be held over for any number of time periods at the source or sink. Notice that, in this formulation, no explicit mention is made of transit times. They are, however, implicitly included in the definition. Thus if edge (x, y) has capacity 10 and transit time 3, this would be indicated by the relation

$$\begin{aligned} \gamma(x_i, y_j) &= 10 \quad \text{for } j = i + 3, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

For the network X_n , the notions of flow, demand, feasible demand, and maximal demand are defined exactly as before.

Finally, for any integer $k \leq n$, we define X_k , the k -stage subnetwork of X_n , to be the two-terminal network $[X_k, s_0, s_k; \gamma]$, where γ is the same function as that for X_n except that it is restricted to pairs (x_i, y_j) with $i, j \leq k$.

Lemma 2. Let δ_k be a maximal demand on X_k and let $\mu_k = \delta_k(s'_k)$. Then $\mu_k \leq \mu_{k+1}$ for all $k < n$.

Proof. Let ϕ_k be the maximal flow such that

$$\mu_k = \delta_k(s'_k) = \sum_{x \in X_k} \phi_k(x, s'_k).$$

Now define a new demand δ' as follows:

$$\delta'(x_1) = \delta_k(x_1) \text{ for } x_1 \in X_k, x_1 \neq s'_k,$$

$$\delta'(s'_{k+1}) = \mu_k,$$

$$\delta'(x_1) = 0 \text{ otherwise.}$$

This demand is feasible, since it is satisfied by choosing the flow ϕ' , where

$$\phi'(x_1, y_j) = \phi_k(x_1, y_j) \text{ for } 1, j \leq k,$$

$$\phi'(s'_k, s'_{k+1}) = -\phi'(s'_{k+1}, s'_k) = \mu_k,$$

$$\phi'(x_1, y_j) = 0 \text{ otherwise.}$$

By definition of a maximal demand, the conclusion of the lemma follows.

Theorem. Let $\delta_1, \dots, \delta_n$ be maximal demands on X_1, \dots, X_n , and let $\mu_1 = \delta_1(s'_1)$. Then the demand δ , where $\delta(s_1) = \mu_1$, $\delta(s'_1) = \mu_1 - \mu_{1-1}$ for $1 > 1$, and $\delta(x) = 0$ otherwise, is feasible.

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In view of Lemma 2, the theorem is, in fact, simply a special case of Lemma 1 and the proof is therefore immediate.

4. REMARKS

(a) The problem of a universal maximal flow makes sense for the case where there is not one, but several sources in the network: s_1, \dots, s_n . The analogous theorem is true in this case; namely, there exists a dynamic flow from the sources s_1, \dots, s_n into s' that is maximal at all times $1 \leq n$. In fact, this case is easily reduced to the case of a single source by the standard device of introducing a new source s_0 into the network and defining the capacities $\gamma(s_0, s_i)$ to be infinite, and of transit time 1.

(b) One might hope that the theorem on universal maximal flows would extend to the case where there is more than one sink. A natural generalization would be: let $\delta_1, \dots, \delta_n$ be feasible demands on X_1, \dots, X_n such that $\delta_1(x_1) \leq \delta_{i+1}(x_{i+1})$ for all x . Can one then prove the feasibility of the demand δ defined by these functions? The defining equations are

$$f(x_1) = \delta_1(x_1),$$

$$\delta(x_i) = \delta_1(x_i) - \delta_{i-1}(x_{i-1}) \quad \text{for } i > 1.$$

The simple example illustrated in Fig. 1 shows that this is not the case. Here the edges (s, x) and (x, y) both have capacities and transit times equal to 1. The graph of X_2 is

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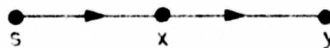


Fig. 1

shown in Fig. 2. Now define δ_1 and δ_2 as follows:

$$\delta_1(x_1) = 1, \quad \delta_1(y_1) = 0,$$

$$\delta_2(x_1) = 0, \quad \delta_2(y_1) = 0, \quad \delta_2(x_2) = 1, \quad \delta_2(y_2) = 1.$$

Clearly, both δ_1 and δ_2 are feasible, but this is not the case for the demand δ defined by

$$\delta(x_1) = 1, \quad \delta(y_1) = 0,$$

$$\delta(x_2) = 0, \quad \delta(y_2) = 1,$$

as the reader will see on referring to Fig. 2.

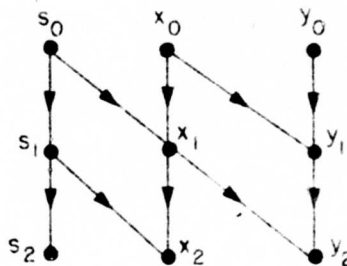


Fig. 2

(c) For the case treated by Ford and Fulkerson, in which capacities are independent of time, these authors showed that a dynamic maximal flow could be achieved that did not involve "holdovers"; in our notation, the maximal flow ϕ_k had the property that $\phi_k(x_i, x_{i+1}) = 0$ for all x_i other than s and s' . In view of their result, the same thing is true for the case of universal maximal flows when capacities are constant with time. Namely, one simply defines the capacity γ so that $\gamma(x_i, x_{i+1}) = 0$ for $x \neq s, s'$, and the proof is exactly as before.

(d) For the case of constant capacities, Ford and Fulkerson have given a simple computational method for finding maximal flows. The proof given here does not lead to such a procedure. We suspect that some mild modification of the Ford-Fulkerson algorithm will actually allow the calculation of universal maximal flows. For the present, however, this is pure conjecture.

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2. Gale, D., "A Theorem on Flows in Networks," Pacific Journal of Mathematics, Vol. 7, No. 2, 1957, pp. 1073-1082.